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On the evolution of a solitary wave for very weak nonlinearity

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The initial-value problem for a one-dimensional gravity wave of amplitude a and characteristic length l in water of depth d is examined for $0 < a/d \ll d^2/l^2 \ll 1$. A preliminary reduction leads to a Korteweg–de Vries (KdV) equation in which the nonlinear term is $O(\epsilon)$ relative to the linear terms, where $\epsilon = 3al^2/4d^3 \ll 1$ is a measure of nonlinearity/dispersion. The linear approximation ($\epsilon \downarrow 0$) is found to be valid if and only if $\epsilon\tau^{\frac{1}{3}} \ll 1$, where $\tau = \frac{1}{2}(d/l)^2 (gd)^{\frac{1}{2}}(\text{time})/l$ is the slow time in the KdV equation. The asymptotic solution of the KdV equation is obtained with the aid of inverse-scattering theory and is found to comprise not only a decaying wave train that is qualitatively similar to that predicted by the linear approximation, but also a soliton of amplitude $3V^2/4d^3 = O(\epsilon a)$ if $V > 0$, where V is the cross-sectional area of the initial displacement, or of amplitude $= O(\epsilon^3 a)$ if $V = 0$ (there is no soliton if $V < 0$). This soliton is fully evolved, and dominates the solution, only for $\epsilon\tau^{\frac{1}{3}} \gg 1$ if $V > 0$ or $\epsilon^2\tau^{\frac{1}{3}} \gg 1$ if $V = 0$, but nonlinearity already has significant effects for $\epsilon\tau^{\frac{1}{3}} = O(1)$.

1. Introduction

Ursell (1953), in his seminal paper on the scaling of nonlinear waves, remarks that an initially localized, *positive* displacement (a mound of water) in a wave tank, no matter how small, ultimately produces a solitary wave (Scott Russell 1844), whereas linear theory, presumably valid in the limit of small amplitudes, yields no such wave. The seeming contradiction, which may appropriately be designated as the *Ursell paradox*, is a consequence of the non-uniform validity of linear theory for sufficiently large times, i.e. of the non-commutativity of the limits *amplitude* $\rightarrow 0$ and *time* $\rightarrow \infty$. I consider here the analytical resolution of that paradox and the description of the soliton for initial displacements of either positive or zero volume.

Consider a homogeneous inviscid liquid of uniform depth d and let a be a characteristic amplitude and l a characteristic wavelength; then nonlinearity and dispersion are measured by

$$\alpha = a/d, \quad \beta = (d/l)^2, \tag{1.1 a, b}$$

respectively, and their relative importance is measured by the parameter (Ursell 1953)

$$\epsilon = 3\alpha/4\beta = 3al^2/4d^3. \tag{1.2}$$

Linear theory describes the limit $\alpha \rightarrow 0$ with β fixed (and hence $\epsilon \rightarrow 0$) on the hypothesis that l and $l(gd)^{-\frac{1}{2}}$ are representative scales of horizontal distance and time. Boussinesq or, in the present context, Korteweg–de Vries (KdV) theory describes the joint limit $\alpha \rightarrow 0$, $\beta \rightarrow 0$ with ϵ fixed and comprises linear long-wave theory through the subsequent limit $\epsilon \to 0$. The Ursell paradox therefore may be investigated by invoking KdV theory and then letting $\epsilon \to 0$.

The solution of the Boussinesq equations (cf. Whitham 1974, § 13.11), which govern the free-surface displacement ay and the depth-averaged horizontal velocity $a(g/d)^{\frac{1}{2}}u$ on the basis of the preceding assumptions, may be posited in the form†

$$y(x,t) = \eta_+(x-t,\tau) + \eta_-(-x-t,\tau) + O(\alpha) \tag{1.3a}$$

nd

$$u(x,t) = \eta_+(x-t,\tau) - \eta_-(-x-t,\tau) + O(\alpha), \tag{1.3b}$$

where lx is the horizontal co-ordinate, $l(gd)^{-\frac{1}{2}}t$ is the time,

$$\tau = \tfrac{1}{2}\beta t \tag{1.4}$$

is a slow time, and $\eta_\pm(\xi,\tau)$ is slowly varying in a reference frame moving to the right/left with the basic speed $(gd)^{\frac{1}{2}}$ and satisfies the Korteweg–de Vries (KdV) equation

$$\eta_\tau + \tfrac{1}{3}\eta_{\xi\xi\xi} + 4\epsilon\eta\eta_\xi = 0 \tag{1.5}$$

and the initial condition

$$\eta_\pm(\xi,0) = \tfrac{1}{2}\{y(\pm\xi,0) \pm u(\pm\xi,0)\} \equiv \eta_{0\pm}(\xi), \tag{1.6}$$

wherein alternative signs and subscripts are vertically ordered. Mass (or volume), momentum and energy, each of which is conserved, are measured by $\langle\eta_+ + \eta_-\rangle$, $\langle\eta_+ - \eta_-\rangle$ and $\langle\eta_+^2 + \eta_-^2\rangle$, respectively, where

$$\langle f(x)\rangle \equiv \int_{-\infty}^{\infty} f(x)\,dx. \tag{1.7}$$

I henceforth omit the subscripts and consider the solution of (1.5) subject to the reduced boundary condition

$$\eta(\xi,0) = \eta_0(\xi). \tag{1.8}$$

The solution of the linearized KdV equation

$$\eta_\tau + \tfrac{1}{3}\eta_{\xi\xi\xi} = 0 \tag{1.9}$$

and the initial condition (1.8) has the asymptotic form (see § 2)

$$\eta(\xi,\tau) \sim \langle\eta_0\rangle\,\tau^{-\frac{1}{3}}\,\mathrm{Ai}\,(\tau^{-\frac{1}{3}}\xi) \quad (\tau\uparrow\infty) \tag{1.10}$$

and is characterized by a steeply rising wave front in $\xi \gtrsim \tau^{\frac{1}{3}}$ and by a slowly decaying, dispersive wave train in $\xi \lesssim -\tau^{\frac{1}{3}}$. It might appear that the asymptotic solution of (1.5) and (1.8) for $0 < \epsilon \ll 1$ could be similarly characterized; however, it is known (Ursell 1953; Segur 1973) that if $\langle\eta_0\rangle > 0$ this solution comprises both a decaying (as $\tau\uparrow\infty$) component, which bears at least some qualitative similarity to that predicted by linear theory, and a soliton of the form

$$\eta_1(\xi,\tau) = (\kappa^2/\epsilon)\,\mathrm{sech}^2\,(\kappa\xi - \tfrac{4}{3}\kappa^3\tau + \delta), \tag{1.11}$$

which is fully evolved only for $\kappa\tau^{\frac{1}{3}} \gg 1$. That there is only one soliton is evident *a priori* from Segur's (1973) extension of Bargmann's inequality, which gives the upper bound

$$N \leqslant 1 + 2\epsilon\int_{-\infty}^{\infty} |\xi|\,q(\xi)\,d\xi, \quad q = {\eta_0 \atop 0}\,(\eta_0 \gtrless 0), \tag{1.12}$$

† The basic argument on which this superposition rests, that the interaction between the oppositely moving waves is weak by virtue of the relatively brief interaction time, goes back to Gwyther (1900) and has since been re-advanced by others; see Miles (1977) for references and a more detailed derivation.

on the number of solitons that evolve from η_0 (the integral is uniformly bounded as $\epsilon \downarrow 0$ by virtue of the assumption of compact support and the definition of l). I obtain (in § 5) the explicit result

$$\kappa = \epsilon\langle\eta_0\rangle + O(\epsilon^2), \quad \delta = \tfrac{1}{2}\ln 2 + O(\epsilon), \qquad (1.13a, b)$$

where, here and subsequently, $\langle\eta_0\rangle = O(1)$ except as noted. [The true amplitude of the soliton described by (1.11) and (1.13) is $3V^2/4d^3$, where V is the cross-sectional area of the initial displacement.] It follows from (1.11) and (1.13) that

$$\langle\eta_1\rangle = 2\langle\eta_0\rangle + O(\epsilon) \qquad (1.14)$$

and hence that the dispersive component of the solution must have a relative mass $-\langle\eta_0\rangle$, rather than the value $\langle\eta_0\rangle$ predicted by linear theory, in order to conserve the initial value $\langle\eta_0\rangle$.

There is no soliton if $\langle\eta_0\rangle < 0$ in the limit $\epsilon \downarrow 0$, but even then scaling considerations (see § 2), which also hold for $\langle\eta_0\rangle > 0$, imply that nonlinearity is significant for $\epsilon\tau^{\frac{1}{3}} = O(1)$ and hence that the linear approximation is valid only for $\epsilon\tau^{\frac{1}{3}} \ll 1$ (Ursell 1953). These same considerations also suggest that a direct perturbation expansion starting from the linear approximation is not uniformly convergent as $\tau \uparrow \infty$.

The reckoning is more delicate if $\langle\eta_0\rangle = 0$, which is a necessary condition if the wave motion is initiated from rest by deforming the free surface from its quiescent level (the state of static equilibrium).† A direct perturbation analysis (see § 2) then appears to succeed and to provide an asymptotic description in the form of a similarity solution (Berezin & Karpman 1964, 1967); in fact, I find that (see § 5) $0 < \epsilon \ll 1$ and $\langle\eta_0\rangle = 0$ imply the existence of a single soliton of the form (1.11) with

$$\kappa = 2\epsilon^2\langle\phi_0^2\rangle + O(\epsilon^3), \quad \phi_0(x) = \int_{\infty \operatorname{sgn} x}^{x} \eta_0(\zeta)\, d\zeta \quad (\langle\eta_0\rangle = 0). \qquad (1.15a, b)$$

It is true that the amplitude and relative mass of η_1 now are $\kappa^2/\epsilon = O(\epsilon^3)$ and $2\kappa/\epsilon = O(\epsilon)$, respectively, and therefore negligible in the limit $\epsilon \downarrow 0$ with τ fixed; nevertheless, η_1 dominates the solution in $\tau \gg 1/\kappa^3$.

I proceed as follows. The linearized solution, including (1.10) and its generalization, and the aforementioned similarity solution are considered in § 2. The inverse-scattering algorithm, which reduces the solution of the KdV equation to the solution of a scattering problem and of a linear (Marchenko) integral equation, is recapitulated in § 3. An integral-equation formulation of the scattering problem is briefly developed in § 4, following established procedures in quantum mechanics. The explicit solution of the scattering problem for $\epsilon \ll 1$, including the derivation of (1.13) and (1.15), is developed in § 5. The resulting approximations are used in § 6 to describe the evolution of the KdV solution from the linear approximation in $\epsilon\tau^{\frac{1}{3}} = O(1)$.

The soliton is distinct from, and dominates, the dispersive component of the solution in $\kappa\tau^{\frac{1}{3}} \gg 1$. A description of the latter component may be obtained, at least in principle, by separating out the soliton to obtain a modified Marchenko integral equation, which admits a convergent Neumann-series solution; however, explicit results appear to be difficult to obtain through this procedure, and I therefore have

† The free-surface displacement in a laboratory wave tank, in which the motion is initiated from a localized mound of water, is typically measured from a depressed level surface that is *not* one of static equilibrium.

relegated it to an appendix. Finally, it should be observed that viscous dissipation could prove more significant than nonlinearity for the fine structure of the dispersive component over the very long time intervals implied by $\kappa\tau^{\frac{1}{3}} \gg 1$.

2. Linear and similarity approximations

The solution of the linear KdV equation (1.9) and the initial condition (1.8) is given by (Berezin & Karpman 1964)

$$\eta(\xi,\tau) = \int_{-\infty}^{\infty} \eta_0(\zeta)\, G(\xi-\zeta,\tau)\, d\zeta \equiv \eta^{(0)}(\xi,\tau), \tag{2.1}$$

where

$$G(\xi,\tau) = \tau^{-\frac{1}{3}}\,\mathrm{Ai}\,(\tau^{-\frac{1}{3}}\xi) \tag{2.2}$$

is the Green's function for (1.9). Letting $\tau \uparrow \infty$ with $\xi = O(\tau^{\frac{1}{3}})$ and invoking the restriction that η_0 be of compact support, we obtain the asymptotic expansion

$$\eta^{(0)}(\xi,\tau) \sim \sum_{n=0}^{\infty} (\langle x^n \eta_0\rangle / n!)\,(-\partial_\xi)^n\, G(\xi,\tau) \tag{2.3a}$$

$$\sim \langle \eta_0\rangle\, \tau^{-\frac{1}{3}}\,\mathrm{Ai}\,(\tau^{-\frac{1}{3}}\xi) - \langle x\eta_0\rangle\, \tau^{-\frac{2}{3}}\,\mathrm{Ai}'\,(\tau^{-\frac{1}{3}}\xi) + \ldots, \tag{2.3b}$$

in which $\langle \eta_0\rangle, \langle x\eta_0\rangle, \ldots$ are the source, dipole, ... moments of the initial distribution.

The linear approximation (2.1) could be extended by regarding the nonlinear term in (1.5) as a perturbation forcing function, constructing the implicit solution of (1.5) and (1.8) in the form

$$\eta(\xi,\tau) = \eta^{(0)}(\xi,\tau) - 2\epsilon\,\partial_\xi \int_{-\infty}^{\infty}\int_0^{\tau} \eta^2(\zeta,\sigma)\, G(\xi-\zeta,\tau-\sigma)\, d\zeta\, d\sigma, \tag{2.4}$$

and solving either by iteration or by expanding η in integral powers of ϵ. It is already evident from (2.3b), however, that the resulting expansion cannot be uniformly valid if $\langle \eta_0\rangle \neq 0$, for then the substitution of the dominant term in the asymptotic expansion into the KdV equation (1.5) with $\xi = O(\tau^{\frac{1}{3}})$ reveals that the linear and nonlinear terms are $O(\tau^{-\frac{4}{3}})$ and $O(\epsilon\tau^{-1})$, respectively, in consequence of which the linear approximation fails for $\tau^{\frac{1}{3}} = O(\epsilon^{-1})$.

The perturbation solution appears more promising if $\langle \eta_0\rangle = 0$, for then the corresponding substitution implies that the linear and nonlinear terms in (1.5) are $O(\tau^{-\frac{5}{3}})$ and $O(\epsilon\tau^{-\frac{5}{3}})$, respectively, and hence that the nonlinear term remains uniformly $O(\epsilon)$ relative to the linear terms. This suggests that the asymptotic solution of (1.5) and (1.8) for $\langle \eta_0\rangle = 0$ should be of the form (Berezin & Karpman 1964)†

$$\epsilon\eta(\xi,\tau) \sim \tfrac{1}{2}\tau^{-\frac{2}{3}}\mathcal{N}(x), \quad x = \tau^{-\frac{1}{3}}\xi \quad (\tau \uparrow \infty), \tag{2.5a, b}$$

where $\mathcal{N}(x)$ satisfies [substitute (2.5) into (1.5)]

$$\mathcal{N}''' - x\mathcal{N}' - 2\mathcal{N} + 6\mathcal{N}\mathcal{N}' = 0. \tag{2.6}$$

† Karpman (1967) claims that no soliton exists if $\epsilon \ll 1$ and $\langle \eta_0\rangle = 0$, whilst Berezin & Karpman (1967) claim that the exact solution of (1.5) and (1.8) for $\eta_0(x) = \delta'(x)$ is of the form (2.5). The former claim is negated by the present results; see (1.15) and §5. The latter claim also is negated for any finite approximation to $\delta'(x)$, but the significance of the limit $\eta_0 \rightarrow \delta'(x)$ is obscured by the prior restriction $l \gg d$.

In fact (see § 6), the asymptotic solution of (1.5) and (1.8) contains a term proportional to $\epsilon\tau^{-\frac{1}{3}}\,\mathrm{Ai}\,(x)$ for $\epsilon\tau^{\frac{1}{3}} = O(1)$, in consequence of which the similarity solution (2.5) offers no real improvement over the linear approximation.

3. The inverse-scattering algorithm

The solution of (1.5) and (1.8) may be carried out according to the following algorithm of Gardner, Greene, Kruskal & Miura [see Whitham 1974, § 17.3, after letting $u = -2\epsilon\eta$, $x = \xi$ and $t = \frac{1}{3}\tau$ therein so that x in this section differs from x in § 1, and $B(x,\tau)$ replaces Whitham's $B(x,t)$].

(i) Solve the scattering problem posed by the one-dimensional Schrödinger equation

$$\{(d/dx)^2 + k^2 + 2\epsilon\eta_0(x)\}\,\psi(x,k) = 0 \quad (-\infty < x < \infty) \tag{3.1}$$

and the radiation condition

$$\psi \sim e^{-ikx} + b(k)\,e^{ikx} \quad (x \uparrow \infty) \tag{3.2}$$

to obtain the reflexion coefficient $b(k)$ over the continuous spectrum $-\infty < k < \infty$.

(ii) Solve the eigenvalue problem posed by (3.1) and

$$\psi \sim e^{-\kappa x} \quad (x \uparrow \infty), \quad \gamma \equiv \langle\psi^2\rangle^{-1} > 0 \quad (k = i\kappa) \tag{3.3a, b}$$

to obtain the discrete eigenvalues $\kappa_1 > \kappa_2 > \ldots > \kappa_N > 0$ ($N \equiv 0$ if the discrete spectrum is empty) and the normalizing parameters γ_n. We exclude *pure-soliton* problems, for which $b(k) \equiv 0$ (see Whitham 1974, § 17.4). The discrete eigenvalues then are given by the poles of $b(k)$ on the *positive* imaginary axis of the complex k plane ['false poles' are ruled out by the restriction that $\eta_0(x)$ be of compact support; see DeAlfaro & Regge 1965, § 7.3]. The corresponding normalizing parameters are given by

$$\gamma_n = -i\,\mathrm{Res}\,\{b(k), k = i\kappa_n\}. \tag{3.4}$$

(iii) Determine the auxiliary function

$$B(x,\tau) = (2\pi)^{-1}\int_{-\infty}^{\infty} b(k)\exp\left(ikx + \tfrac{8}{3}ik^3\tau\right)dk + \sum_{n=1}^{N}\gamma_n\exp\left(-\kappa_n x + \tfrac{8}{3}\kappa_n^3\tau\right). \tag{3.5}$$

(iv) Solve the Marchenko equation

$$K(x,y,\tau) + B(x+y,\tau) + \int_x^{\infty} B(y+z,\tau)\,K(x,z,\tau)\,dz = 0 \quad (y > x). \tag{3.6}$$

(v) The required solution of (1.5) and (1.8) then is given by

$$\epsilon\eta(\xi,\tau) = \partial_\xi K(\xi,\xi,\tau). \tag{3.7}$$

The argument τ evidently enters the inverse-scattering algorithm only as a parameter and is henceforth implicit in both B and K if not explicitly displayed.

4. Integral-equation formulation of the scattering problem†

The solution of (3.1) may be posed in the implicit form

$$\psi(x,k) = e^{-ikx} - \epsilon(ik)^{-1}\int_{-\infty}^{\infty} e^{ik|x-y|}\psi(y,k)\,\eta_0(y)\,dy, \tag{4.1}$$

in which the first and second terms represent the incident and scattered waves. Letting $x\uparrow\infty$ in (4.1) and comparing the result to (3.2), we obtain

$$b(k) = -\epsilon(ik)^{-1}\int_{-\infty}^{\infty} e^{-iky}\psi(y,k)\,\eta_0(y)\,dy. \tag{4.2}$$

Combining (4.1) and (4.2) or, alternatively, solving (3.1) and (3.2) by variation of parameters, we obtain the equivalent integral equation

$$\psi(x,k) = e^{-ikx} + b(k)\,e^{ikx} + 2\epsilon k^{-1}\int_{x}^{\infty} \sin\{k(x-y)\}\,\psi(y,k)\,\eta_0(y)\,dy. \tag{4.3}$$

The integral equation (4.3) admits a solution of the form

$$\psi(x,k) = f(x,k) + b(k)f(x,-k), \tag{4.4}$$

where $f(x,k)$ is determined by the reduced integral equation

$$f(x,k) = e^{-ikx} + 2\epsilon k^{-1}\int_{x}^{\infty} \sin\{k(x-y)\}f(y,k)\,\eta_0(y)\,dy \tag{4.5}$$

or, equivalently, by

$$\{(d/dx)^2 + k^2 + 2\epsilon\eta_0(x)\}f(x,k) = 0 \tag{4.6}$$

and

$$f(x,k) \sim e^{-ikx} \quad (x\uparrow\infty). \tag{4.7}$$

Substituting (4.4) into (4.2), we obtain

$$b(k) = -\epsilon\left\{ik + \epsilon\int_{-\infty}^{\infty} e^{-ikx}f(x,-k)\,\eta_0(x)\,dx\right\}^{-1}\int_{-\infty}^{\infty} e^{-ikx}f(x,k)\,\eta_0(x)\,dx. \tag{4.8}$$

The one-dimensional scattering problem differs from its spherically symmetric counterpart both in the range of x (or r), $(-\infty,\infty)$ *vs.* $(0,\infty)$, and in the fact that regularity at the origin requires $\psi(0,k) = 0$ (if $r\psi$ is the wave function) for the latter problem. The analogy between the two problems is much closer if $\eta_0(x)$ is even: then only $(0,\infty)$ need be considered, and the extension of Jost's formulation (Goldberger & Watson 1964) yields

$$b(k) = -\frac{1}{2}\left\{\frac{f(0,k)}{f(0,-k)} + \frac{f'(0,k)}{f'(0,-k)}\right\}, \tag{4.9}$$

where $f'(0,k) \equiv df(x,k)/dx$ at $x = 0$.

† The development in this section is analogous to that given in quantum-mechanics textbooks, e.g. Goldberger & Watson (1964), for spherically symmetric scattering under the rubric of the Jost function; see also DeAlfaro & Regge (1965).

5. Solution of the scattering problem for $\epsilon \ll 1$

The integral equation (4.5) may be solved by iteration or, equivalently, by expanding f in powers of ϵ; convergence can be proved (cf. Goldberger & Watson 1964, p. 272) and is uniform by virtue of the restriction that $\eta_0(x)$ be of compact support. Substituting the first approximation

$$f(x,k) = e^{-ikx}\{1+O(\epsilon)\} \tag{5.1}$$

into (4.8), we obtain

$$b(k) = -\epsilon\{ik+\epsilon N(0)\}^{-1} N(2k), \tag{5.2}$$

where

$$N(k) = \int_{-\infty}^{\infty} e^{-ikx}\eta_0(x)\,dx \tag{5.3}$$

is the Fourier transform of $\eta_0(x)$, and $O(\epsilon^2)$ errors are implicit in both the numerator and the denominator of (5.2). There is a single discrete eigenvalue,

$$\kappa = \epsilon N(0) = \epsilon\langle\eta_0\rangle, \tag{5.4}$$

if $\langle\eta_0\rangle > 0$. The corresponding approximation to the residue of $-ib$ at $k = i\kappa$ is $\gamma = \kappa$.

The approximation (5.2), which implies the existence of a single soliton if $\langle\eta_0\rangle > 0$, is uniformly valid for all k as $\epsilon \downarrow 0$ if $\langle\eta_0\rangle = O(1)$. It is exact for $\eta_0 = \pm\delta(x)$, but this example is of rather dubious significance in the present context, in which (from the definitions of the length scales a and l) η_0 is assumed to be of order unity over an x interval of order unity. A more significant example is $\eta_0 = \operatorname{sech}^2 x$, for which the exact solution of the scattering problem yields $\kappa_1 = \frac{1}{2}\{(1+8\epsilon)^{\frac{1}{2}}-1\}$ and $N = 1$ for $0 < \epsilon \leqslant 1$; the limiting approximation $\kappa_1 = 2$ $(\epsilon \downarrow 0)$ agrees with (5.4) ($\eta_0 = \operatorname{sech}^2 x$ is not of compact support, but its exponential decay as $|x| \to \infty$ is a sufficient alternative in the present context).

The second approximation to the solution of (4.5) is given by

$$e^{ikx}f(x,k) = 1+\epsilon(ik)^{-1}\int_{x}^{\infty}\{e^{2ik(x-y)}-1\}\,\eta_0(y)\,dy+O(\epsilon^2). \tag{5.5}$$

Substituting (5.5) into (4.8) and transforming the resulting double integrals to symmetric forms, we obtain

$$-b(k) = \frac{\epsilon N(2k)+\epsilon^2(2ik)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{-2ikx}-e^{-2iky})\,\eta_0(x)\,\eta_0(y)\operatorname{sgn}(x-y)\,dy\,dx}{ik+\epsilon N(0)-\epsilon^2(2ik)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{2ik|x-y|}-1)\,\eta_0(x)\,\eta_0(y)\,dy\,dx}, \tag{5.6}$$

in which $N(k)$ is given by (5.3) and $O(\epsilon^3)$ errors are implicit in both numerator and denominator. Expanding the numerator and denominator about $k = 0$, simplifying the remaining integrals through integration by parts, and introducing

$$\phi_0(x) = \int_{\infty\,\operatorname{sgn}x}^{x}\eta_0(\zeta)\,d\zeta, \quad \phi_0'(x) = \eta_0(x)-\langle\eta_0\rangle\,\delta(x), \tag{5.7 a, b}$$

we obtain

$$b(k) = -(\kappa+ik)^{-1}\{\kappa+2\lambda ik+O(\epsilon k^2, \epsilon^2 k)\}, \tag{5.8}$$

where

$$\kappa = \epsilon\langle\eta_0\rangle+2\epsilon^2(\langle\phi_0^2\rangle-\langle\eta_0\rangle\langle|x|\eta_0\rangle)+O(\epsilon^3) \tag{5.9}$$

and

$$\lambda = -\epsilon\langle x\eta_0\rangle = \epsilon\langle\phi_0\rangle. \tag{5.10}$$

The corresponding approximation to the residue of $-ib$ at $k = i\kappa$ is

$$\gamma = \kappa(1-2\lambda)+O(\epsilon^3). \tag{5.11}$$

It follows from (5.8) that $b(0) = -1$ unless $\kappa = 0$, i.e. unless $\langle\eta_0\rangle = -2\epsilon\langle\phi_0^2\rangle+O(\epsilon^2)$. It follows from (5.9) that there is one and only one soliton for an initial displacement of zero volume ($\langle\eta_0\rangle = 0$) in the limit $\epsilon \downarrow 0$.

The second approximation (5.6) differs only quantitatively from the first approximation (5.2) if $\langle\eta_0\rangle = O(1)$ as $\epsilon \downarrow 0$; but it differs qualitatively, and is uniformly valid, near $k = 0$ if $\langle\eta_0\rangle = 0$ [or, more generally, $\langle\eta_0\rangle = O(\epsilon)$]. A simpler approximation, which retains this uniform validity and reduces to (5.8) for $k \downarrow 0$, is [cf. (5.2)]

$$b(k) = -(ik+\kappa)^{-1}\{\epsilon N(2k)+\kappa-\epsilon\langle\eta_0\rangle\}+O(\epsilon^2), \tag{5.12}$$

where κ is given by (5.9).

6. Evolution of η in $\epsilon\tau^{\frac{1}{3}} = (O)1$

The evolution of η from $\xi,\tau = O(1)$, where the linear approximation (2.1) is valid for $\epsilon \ll 1$, into $\epsilon\tau^{\frac{1}{3}} = O(1)$, where nonlinear terms in the solution of the KdV equation are comparable with the linear term, may be inferred from the Neumann-series solution (Segur 1973)

$$\epsilon\eta(\xi,\tau) = -\partial_\xi B(2\xi)-2B^2(2\xi)+2B(2\xi)\int_{2\xi}^{\infty}B^2(z)\,dz \\ -\int_{2\xi}^{\infty}\int_{2\xi}^{\infty}B(z_1)\,B'(z_2)\,B(z_1+z_2-2\xi)\,dz_1dz_2+\dots \tag{6.1}$$

of (3.6) and (3.7). The series is convergent for all ξ and τ if $N = 0$; the proof is given by Miura (1976) under restrictions that are satisfied here through the restriction that $\eta_0(x)$ must be of compact support. The series is not convergent for all ξ and τ if $N > 0$; however, guided by the form of the contributions from the soliton component of $B(\xi,\tau)$, *v.i.*, I conjecture that (6.1) is convergent in $\xi > \frac{4}{3}\kappa^2\tau$ if $N = 1$.

Substituting (5.12), $\gamma_1 = \gamma$ and $\kappa_1 = \kappa$ (if $\kappa > 0$) into (3.5) and evaluating the Fourier integral with the aid of the convolution theorem, we obtain

$$B(2\xi) = \gamma H(\kappa)\exp(-2\kappa\xi+\tfrac{8}{3}\kappa^3\tau)-\int_{-\infty\,\mathrm{sgn}\,\kappa}^{\xi}\exp[-2\kappa(\xi-\zeta)]\{\epsilon\eta^{(0)}(\zeta,\tau) \\ +(\kappa-\epsilon\langle\eta_0\rangle)\,G(\zeta,\tau)\}\,d\zeta, \tag{6.2}$$

where κ, γ, $\eta^{(0)}$ and G are given by (5.9), (5.11), (2.1) and (2.2), and $H(\kappa)$ is Heaviside's step function. It can be shown that the implicit remainder is $O(\epsilon^2)$ if $\xi,\tau = O(1)$ or $O(\epsilon^3)$ if $\xi,\tau^{\frac{1}{3}} = O(1/\epsilon)$.

Substituting (6.2) into (6.1) and letting $\epsilon \downarrow 0$ with $\xi,\tau = O(1)$, we recover (2.1). The higher-order terms make contributions that are $O(\eta^{(0)})$ for $\tau^{\frac{1}{3}} = O(1/\epsilon)$, in consequence of which (2.1) is valid if and only if $\epsilon\tau^{\frac{1}{3}} \ll 1$, as anticipated in § 2.

Substituting (2.3) into (6.2) and letting $\epsilon \downarrow 0$ with $\tau^{-\frac{1}{3}} = O(\epsilon)$, or, alternatively, substituting (5.8) into (3.5), we obtain (after some reduction)

$$B(2\xi) = \kappa(1-2\lambda)\,F(\xi,\tau)-\lambda G(\xi,\tau)+O(\epsilon^3) \quad [\tau^{-\frac{1}{3}} = O(\epsilon)], \tag{6.3}$$

where

$$F(\xi,\tau) = H(\kappa)\exp\left(-2\kappa\xi+\tfrac{8}{3}\kappa^3\tau\right) - \int_{-\infty\,\mathrm{sgn}\,\kappa}^{\xi} \exp\left[-2\kappa(\xi-\zeta)\right] G(\zeta,\tau)\,d\zeta \qquad (6.4)$$

and λ is given by (5.10). We remark that $\lambda = 0$ and $B(2\xi) = \kappa F(\xi,\tau)$ is exact for $\eta_0(x) = \pm\delta(x)$.

The substitution of (6.3) into (6.1) provides a formal description of the evolution of η in $1 \ll \tau^{\frac{1}{3}} = O(\epsilon^{-1})$; however, each term in the Neumann series is $O(\epsilon^2)$ for $\tau^{-\frac{1}{3}} = O(\epsilon)$ unless $\langle\eta_0\rangle = 0$. In that important special case, $B = O(\epsilon^2)$, the successive terms in (6.1) are $O(\epsilon^3, \epsilon^4, \epsilon^5, \ldots)$, and

$$\eta(\xi,\tau) = 2\epsilon\langle\phi_0^2\rangle\tau^{-\frac{1}{3}}\,\mathrm{Ai}\,(\tau^{-\frac{1}{3}}\xi) + \langle\phi_0\rangle\tau^{-\frac{2}{3}}\,\mathrm{Ai}'\,(\tau^{-\frac{1}{3}}\xi) + O(\epsilon^3) \quad (\langle\eta_0\rangle = 0). \qquad (6.5)$$

The similarity solution (2.5) comprises the second but not the first term in (6.5) and therefore is valid only in $\epsilon\tau^{\frac{1}{3}} \ll 1$, the domain of the linear approximation. We note that (6.5) does not include the precursor of the soliton, which is $O(\epsilon^3)$ if $\langle\eta_0\rangle = 0$.

$B(2\xi,\tau)$ is exponentially small, and (6.1) converges rapidly, if $\kappa\xi - \frac{4}{3}\kappa^3\tau \gg 1$ (or $\xi \gg \tau^{\frac{1}{3}}$ if $\kappa < 0$). The path of integration for the Fourier integral in (3.5) then may be deformed into the path of steepest descent through the saddle point at $k = \frac{1}{2}i(\xi/\tau)^{\frac{1}{2}}$, which lies below/above the pole at $k = i\kappa$ (for $\kappa > 0$) if $\xi \lessgtr 4\kappa^2\tau$. Carrying out the saddle-point approximation and allowing for the contribution of the pole, which just cancels the contribution of the discrete spectrum if $\xi > 4\kappa^2\tau$, we obtain

$$B(2\xi) \sim \gamma H(\kappa)\,H(4\kappa^2\tau-\xi)\exp\left(-2\kappa\xi+\tfrac{8}{3}\kappa^3\tau\right) + \tfrac{1}{4}\pi^{-\frac{1}{2}}b\{\tfrac{1}{2}i(\xi/\tau)^{\frac{1}{2}}\}\,(\xi\tau)^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}\tau^{-\frac{1}{2}}\xi^{\frac{3}{2}}\right). \qquad (6.6)$$

An extension of (6.6) that is uniformly valid in the neighbourhood of $\xi = 4\kappa^2\tau$ may be obtained by separating out the singular part of b and transforming the saddle-point approximation to the singular integral to that error-function integral denoted as $w(z)$ by Abramowitz & Stegun (1964, § 7.1.3).

Substituting (6.6) into (6.1), we obtain

$$\epsilon\eta(\xi,\tau) \sim 2\gamma\kappa H(\kappa)\,H(4\kappa^2\tau-\xi)\exp\left(-2\kappa\xi+\tfrac{8}{3}\kappa^3\tau\right) + \tfrac{1}{4}\pi^{-\frac{1}{2}}b\{\tfrac{1}{2}i(\xi/\tau)^{\frac{1}{2}}\}\,\tau^{-\frac{3}{4}}\xi^{\frac{1}{4}}\exp\left(-\tfrac{2}{3}\tau^{-\frac{1}{2}}\xi^{\frac{3}{2}}\right), \qquad (6.7)$$

which extends Ablowitz & Newell's (1973) result for $N = 0$ to $N = 1$. Similar results may be obtained for $N > 1$ by allowing for the additional poles; in particular, $\epsilon\eta$ is asymptotic to the second term in (6.7) for $\xi > 4\kappa_1^2\tau$, where κ_1 is the largest eigenvalue.

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Appendix. Reduction of Marchenko integral equation for $N = 1, \kappa\tau^{\frac{1}{3}} \uparrow \infty$

We consider here the asymptotic domain $\kappa\tau^{\frac{1}{3}} \gg 1$, in which the soliton is fully developed, for $N = 1$ (for which $\epsilon \ll 1$ and $\langle\eta_0\rangle \geqslant 0$ are sufficient but not necessary conditions). The asymptotic solution for $N = 0$ (for which $\epsilon \ll 1$ and $\langle\eta_0\rangle < 0$ are sufficient) has been considered by Zakharov & Manakov (1976) and Ablowitz & Segur (1977).

Guided by the known solution of (3.6) for the pure-soliton problem and by the asymptotic scaling implicit in the preceding development, we introduce

$$X = \kappa x - \tfrac{4}{3}\kappa^3\tau + \delta, \quad \delta = \tfrac{1}{2}\ln(2\kappa/\gamma), \quad \mathscr{x} = \tfrac{1}{2}\tau^{-\frac{1}{3}}x, \quad \theta = 2\kappa\tau^{\frac{1}{3}}, \qquad \text{(A 1}a\text{–}d)$$

recast (3.5) in the form

$$B(x+y) = 2\kappa e^{-X-Y} + \tfrac{1}{2}\tau^{-\frac{1}{3}}\mathscr{B}(\mathscr{x}+\mathscr{y}), \qquad \text{(A 2}a)$$

where
$$\mathscr{B}(\mathscr{x}) = (2\pi)^{-1}\int_{-\infty}^{\infty} b(\kappa\mathscr{k}/\theta)\exp[i(\mathscr{k}\mathscr{x}+\tfrac{1}{3}\mathscr{k}^3)]\,d\mathscr{k} \quad (\mathscr{k} = 2k\tau^{\frac{1}{3}}), \qquad \text{(A 2}b)$$

and posit the solution of (3.6) in the form

$$K(x,y) = \kappa\mathsf{K}(X)\,e^{X-Y} + \tfrac{1}{2}\tau^{-\frac{1}{3}}\mathscr{K}(\mathscr{x},\mathscr{y}), \qquad \text{(A 3)}$$

where Y and $\mathscr{y}$ are the counterparts of X and $\mathscr{x}$. We remark that θ is the ratio of the length scales of $\mathscr{x}$ and X ($\partial X/\partial\mathscr{x} = \theta$) and that (A 3) and the subsequent development up to (A 8) is valid for all θ (in particular, it does not neglect the interaction between the soliton and the decaying wave train).

Substituting (A 1), (A 2a) and (A 3) into (3.6), integrating the product of the exponential terms and dividing the result by κ, we obtain

$$e^{X-Y}\{(1+e^{-2X})\,\mathsf{K}(X)+2e^{-2X}\}+\theta^{-1}\left\{\mathscr{K}(\mathscr{x},\mathscr{y})+\mathscr{B}(\mathscr{x}+\mathscr{y})+\int_{\mathscr{x}}^{\infty}\mathscr{B}(\mathscr{y}+\mathscr{z})\,\mathscr{K}(\mathscr{x},\mathscr{z})\,d\mathscr{z}\right\}$$
$$+\int_{\mathscr{x}}^{\infty} e^{\theta(\mathscr{x}-\mathscr{z})}\{2e^{-X-Y}\,\mathscr{K}(\mathscr{x},\mathscr{z})+\mathsf{K}(X)\,\mathscr{B}(\mathscr{y}+\mathscr{z})\}\,d\mathscr{z} = 0 \quad (Y > X, \mathscr{y} > \mathscr{x}). \qquad \text{(A 4)}$$

Requiring the coefficient of $\exp(-Y)$ in (A 4) to vanish separately and invoking the operational identity

$$\int_{\mathscr{x}}^{\infty} e^{\theta(\mathscr{x}-\mathscr{z})}\,\mathscr{B}(\mathscr{y}+\mathscr{z})\,d\mathscr{z} \equiv (\theta-\partial_{\mathscr{x}})^{-1}\,\mathscr{B}(\mathscr{x}+\mathscr{y}), \qquad \text{(A 5)}$$

we obtain
$$\mathsf{K}(X) = \{1+(\theta-\partial_{\mathscr{y}})^{-1}\,\mathscr{K}(\mathscr{x},\mathscr{y})\}_{\mathscr{y}=\mathscr{x}}(\tanh X - 1) \qquad \text{(A 6}a)$$

and
$$\mathscr{K}(\mathscr{x},\mathscr{y})+\{1+\mathsf{K}(X)\,\theta(\theta-\partial_{\mathscr{x}})^{-1}\}\,\mathscr{B}(\mathscr{x}+\mathscr{y})+\int_{\mathscr{x}}^{\infty}\mathscr{B}(\mathscr{y}+\mathscr{z})\mathscr{K}(\mathscr{x},\mathscr{z})\,d\mathscr{z} = 0. \qquad \text{(A 6}b)$$

Substituting (A 6a) into (A 3), setting $x = y = \xi$ and invoking (3.7), we obtain (after some reduction)

$$\epsilon\eta(\xi,\tau) = \kappa^2\operatorname{sech}^2(\kappa\xi-\tfrac{4}{3}\kappa^3\tau+\delta)+\tfrac{1}{2}\tau^{-\frac{2}{3}}\,\mathscr{N}(\tau^{-\frac{1}{3}}\xi) \qquad \text{(A 7)}$$

where
$$\mathscr{N}(2\mathscr{x}) = \tfrac{1}{2}\partial_{\mathscr{x}}\mathscr{K}_*(\mathscr{x},\mathscr{x}), \quad \mathscr{K}_*(\mathscr{x},\mathscr{y}) = \mathscr{J}(X,\partial_{\mathscr{y}})\,\mathscr{K}(\mathscr{x},\mathscr{y}) \qquad \text{(A 8}a, b)$$

and
$$\mathscr{J}(X,\partial_{\mathscr{x}}) = (\theta\tanh X-\partial_{\mathscr{x}})\,(\theta-\partial_{\mathscr{x}})^{-1}. \qquad \text{(A 9)}$$

Note that X enters (A 6b) only as a parameter but that $\partial X/\partial\mathscr{x} = \theta$ in (A 8a).

Letting $\theta\uparrow\infty$ in (A 6a) with $\mathscr{x} = O(\theta)$, we obtain

$$\mathsf{K}(X) \sim (\tanh X - 1)\{1+O(\theta^{-1})\}. \qquad \text{(A 10)}$$

Substituting (A 10) into (A 6b) and introducing

$$\mathscr{B}_*(\mathscr{x},X) \equiv \mathscr{J}(X,\partial_{\mathscr{x}})\,\mathscr{B}(\mathscr{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\theta\tanh X - i\mathscr{k}}{\theta - i\mathscr{k}}\right)b\left(\frac{\kappa\mathscr{k}}{\theta}\right)\exp[i(\mathscr{k}\mathscr{x}+\tfrac{1}{3}\mathscr{k}^3)]\,d\mathscr{k}, \qquad \text{(A 11)}$$

we obtain the modified Marchenko equation

$$\mathscr{K}(x,y)+\mathscr{B}_*(x+y,X)+\int_x^\infty \mathscr{B}(y+z)\mathscr{K}(x,z)\,dz = 0 \quad (y > x), \tag{A 12}$$

which determines $\mathscr{K}$ within $1+O(\theta^{-1})$. The first and second terms in (A 7) now (for $\theta \gg 1$) are distinct and describe the soliton and the decaying wave train.

The integral equation (A 12) differs from its counterpart for $N = 0$ only in the inhomogeneous term ($\mathscr{B}_*$ in place of $\mathscr{B}$) and is solvable in principle through a convergent Neumann expansion.

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